Differential inequality of the second derivative that leads to normality

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Abstract

Let $\mathcal{F}$ be a family of functions meromorphic in a domain D. If $\{\frac{|f''|}{1+|f|^3} : f \in \mathcal{F}\}$ is locally uniformly bounded away from zero, then $\mathcal{F}$ is normal.

I. Introduction.

Recently, progress was occurred concerning the study of the connection between differential inequalities and normality. A natural point of departure for this subject is the well-known theorem due to F.Marty.

Marty's Theorem [8, P.75] A family $\mathcal{F}$ of functions meromorphic in a domain D is normal if and only if $\{f^{\#} : f \in \mathcal{F}\}$ is locally uniformly bounded in D .

Following Marty's Theorem, L. Royden proved the followiing generalization.

Theorem R[7] Let $\mathcal{F}$ be a family of functions meromorphic in a domain D, with the property that for each compact set $K \subset D$, there is a positive increasing function h_K , such that $|f'(z)| \leq h_K(|f(z)|)$ for all $f \in \mathcal{F}$ and $z \in K$. Then $\mathcal{F}$ is normal in D.

This result was significantly extended further in various directions, see $[3], [9]\ and\ [11]$. S.Y.Li and H.Xie established a different kind of generalization of Marty's Theorem that involves higher derivatives.

2010 Mathematical subject classification 30A10, 30D45.

keywords and phrases, Normal family, Differential inequality.

Theorem LX $[4]$ Let $\mathcal{F}$ be a family of functions meromorphic in a domain D, such that each $f \in \mathcal{F}$ has zeros only of multiplicities $\geq k$, $k \in N$. Then $\mathcal{F}$ is normal in D if and only if the family

$$\left\{\frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}} : f \in \mathcal{F}\right\}$$

is locally uniformly bounded in D.

In $[6]$, the second and the third authors gave a counterexample to the validity of Theorem LX, without the condition on the multiplicities of zeros for the case $k = 2$.

Concerning differential inequalities with the reversed sign of the inequality, J. Grahl, and the second author proved the following result, that may be considered as a counterpart to Marty's Theorem.

Theorem GN $[1]$ Let $\mathcal{F}$ be a family of functions meromorphic in D, and $c > 0$. If $f^{\#}(z) > c$ for every $f \in \mathcal{F}$ and $z \in D$, then $\mathcal{F}$ is normal in D.

N.Steinmetz $[10]$, gave a shorter proof of Theorem GN, using the Schwarzian derivative and some Well-known facts on linear differential equations.

Then in $[5]$, X.J.Liu together with the second and third authors generalized Theorem GN and proved the following result.

Theorem LNP Let $1 \leq \alpha < \infty$ and $c > 0$. Let $\mathcal{F}$ be the family of all meroforphic functions f in D, such that

$$\frac{|f'(z)|}{1+|f(z)|^{\alpha}} > C$$

for every $z \in D$.

Then the following hold:

(1) If $\alpha > 1$, then $\mathcal{F}$ is normal in D.

(2) If $\alpha = 1$, then $\mathcal{F}$ is quasi-normal in D but not necessarily normal.

Observe that (2) of the theorem is a differential inequalities that distinguish between quasi-normality to normality.

In this paper, we continue to study differential inequality with the reversed sign $(\text{“} \geq \text{”})$ and prove the following theorem.

Theorem 1. Let D be a domain in $\mathbb{C}$ and let $c > 0$. Then the family $\mathcal{F}$ of all functions f

meromorphic in D , such that

$$\frac{|f''(z)|}{1+|f(z)|^3} > C$$

for every $z \in D$ is normal.

Observe that the above differential inequality is the reversed inequality to that of Theorem LX in the case $k = 2$.

Let us set some notation.

For $z_0 \in C$ and $r > 0$. $\Delta(z_0, r) = \{z : |z - z_0| < r\}$, $\overline{\Delta}(z_0, r) = \{z : |z - z_0| \le r\}$. We write $f_n(z) \overset{\chi}{\Rightarrow} f(z)$ on D to indicate that the sequence $\{f_n(z)\}$ converges to $f(z)$ in the spherical metric, uniformly on compact subsets of D, and $f_n(z) \Rightarrow f(z)$ on D if the convergence is also in the Euclidean metric.

II Proof of Theorem 1.

Since $|f''| > c$ for every $f \in \mathcal{F}$, it follows that $\{f'' : f \in \mathcal{F}\}$ is normal in D . Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions from $\mathcal{F}$. Without loss of generality, we can assume that $f_n''(z) \overset{\chi}{\Rightarrow} H$ in D .

Let us separate into two cases.

Case 1. $f_n, n \ge 1$ are holomorphic functions in D .

Case 1.1 H is holomorphic function in D .

Since normality is a local property. It is enough to prove that $\{f_n\}$ is normal at each point of D . Let $z_0 \in D$ without loss of generality, we can assume that $z_0 = 0$ .By the assumption on H , there exist some $r > 0$, $M > C$, such that $|f_n''(z)| \le M$ for every $z \in \Delta(0, r)$ if n is large enough. We then get for large enough n and $z \in \Delta(0, r)$ that $1 + |f_n(z)|^3 \le \frac{2M}{C}$ and we deduce that $\{f_n\}_{n=1}^{\infty}$ is normal at $z = 0$, as required.

Case 1.2 $H \equiv \infty$ in D.

Again, let $z_0 \in D$ and assume that $z_0 = 0$. Let $r > 0$ be such that $\overline{\Delta}(0, r) \subset D$. Without loss of generality, we can assume that $|f_n''(z)| > 1$ for every $z \in \Delta(0, r)$, $n \in \mathbb{N}$. Then $\log |f_n''|$ is a positive harmonic function in $\Delta(0, r)$.

From Harnack's inequality we then get that

(1)
$$|f_n''(z)| \le |f_n''(0)|^{\frac{1+|z|}{1-|z|}}$$
for every $z \in \Delta(0,r),\ n \in \mathbb{N}$.

Let us fix some $0 < \rho < \dfrac{r}{2}$. Then

(2)
$$\frac{r+\rho}{r-\rho} < 3.$$

For every $n \ge 1$, let $z_n \in \{z : |z| = \rho\}$ be such that
$$|f_n(z_n)| = \max_{|z|\le\rho} |f_n(z)| = M(\rho, f_n)$$

By Cauchy's Inequality , we get that
$$|f_n''(0)| \le \frac{2}{\rho^2} M(\rho, f_n) = \frac{2}{\rho^2}|f_n(z_n)|.$$

Hence, by (1), we get
$$C \le \frac{|f_n''(z_n)|}{1+|f_n(z_n)|^3} \le \frac{|f_n''(z_n)|}{|f_n(z_n)|^3} \le \frac{|f_n''(0)|^{\dfrac{r+\rho}{r-\rho}}}{|f_n(z_n)|^3} \le \left(\frac{2}{\rho^2}\right)^{\dfrac{r+\rho}{r-\rho}} |f_n(z_n)|^{\dfrac{r+\rho}{r-\rho} - 3},$$

Thus, by (2)
$$M(\rho, f_n) = |f(z_n)| \le \left(\frac{1}{C} \left(\frac{2}{\rho^2}\right)^{\dfrac{r+\rho}{r-\rho}} \right)^{\dfrac{1}{3 - \dfrac{r+\rho}{r-\rho}}},$$

which means that $\{f_n\}$ is locally uniformly bounded in $\Delta(0, \rho)$ and thus $\{f_n\}$ is normal at $z = 0$.

Case 2 $\quad f_n$ are meromorphic functions with pole in D.

By Case 1 we have to prove normality only at point z_0, where $H(z_0) = \infty$. Such points exist if H is a meromorphic function with poles in D or if $H \equiv \infty$. So let z_0 be such that $H(z_0) \equiv \infty$. Without loss of generality , we can assume that $z_0 = 0$. After moving to a subsequence, that without loss of generality will also be denoted by $\{f_n\}_1^\infty$, we can assume that there is a sequence $\zeta_n \to 0$ such

that $f_n(\zeta_n)=\infty$. For if it was not the case,then for some $\delta>0$ and large enough n ,f_n would be holomorphic in $\Delta(0,\delta)$,and then we would get the asserted normality by case (1).

Also we can assume the existence of

(3) a sequence $\eta_n \to 0$ such that $f_n(\eta_n)=0$.

Indeed, since $H(z_0)=\infty$ there exists some $\delta>0$ such that for large enough n $\min\limits_{z\in\Delta(0,\delta)}|f_n^{''}|>1$. Combining it with $f_n\neq 0$ in some neighbourhood of $z=0$ gives the normality at $z=0$ by Gu's Criterion [2].

We can also assume that $\{f_n^{'}\}$ is not normal at $z=0$. Indeed, if $\{f_n^{'}\}$ would be normal at $z=0$, then by Marty's theorem there exist $r_1>0$ and $M>0$ such that for large enough n, $\frac{|f_n^{''}(z)|}{1+|f_n^{'}(z)|^2}<M$ for $z\in\Delta(0,r_1)$. Since $H(0)=\infty$,there exists some $r_2\leq r_1$ such that for large enough n, $|f_n^{''}(z)|\geq 2M$ for $z\in\Delta(0,r_2)$.

We thus have for large enough n and $z\in\Delta(0,r_2)$, $1+|f_n^{'}(z)|^2>\frac{|f_n^{''}(z)|}{M}\geq 2$ and thus $|f_n^{'}(z)|\geq 1$.We then get

$$\frac{|f_n^{'}(z)|^2}{|f_n^{''}(z)|}=\frac{|f_n^{'}(z)|^2}{1+|f_n^{'}(z)|^2}\cdot\frac{1+|f_n^{'}(z)|^2}{|f_n^{''}(z)|}\geq\frac{1^2}{1+1^2}\cdot\frac{1}{M}=\frac{1}{2M}.$$

Hence We have for large enough n and $z\in\Delta(0,r_2)$

(4) $$\frac{|f_n^{'}(z)|^2}{1+|f_n(z)|^3}=\frac{|f_n^{'}(z)|^2}{|f_n^{''}(z)|}\cdot\frac{|f_n^{''}(z)|}{1+|f_n(z)|^3}>\frac{1}{2M}\cdot C.$$

Now,for every $x\geq 0$, $\frac{\sqrt{1+x^2}}{1+x}\geq\frac{1}{\sqrt{2}}$, and by taking square root of (4), we get

$$\frac{|f_n^{'}(z)|}{1+|f_n(z)|^{\frac{3}{2}}}=\frac{|f_n^{'}(z)|}{\sqrt{1+|f_n(z)|^3}}\cdot\frac{\sqrt{1+|f_n(z)|^3}}{1+|f_n(z)|^{\frac{3}{2}}}>\sqrt{\frac{C}{2M}}\cdot\frac{1}{\sqrt{2}}.$$

By (1) of Theorem LNP, with $\alpha=\frac{3}{2}>1$, we deduce that $\{f_n\}$ is normal in $\Delta(0,r_2)$ and we are done.

Thus we can assume that $\{f_n'\}$ is not normal at $z=0$.

Similarly to (3) we can assume that there is a sequence $s_n\to 0$ such that $f_n^{'}(s_n)=0$.

We claim that we can assume that $\{\frac{f_n^{'}}{f_n^{''}}\}_{n=1}^{\infty}$ is not normal at $z=0$.

Otherwise, after moving to a subsequence that will also be denoted by $\{\frac{f_n'}{f_n''}\}_{n=1}^{\infty}$ we have $\frac{f_n'}{f_n''} \Rightarrow H_1$ in $\Delta(0,r)$, for some $r > 0$. Since $f_n'' \neq 0$ and $\frac{f_n'}{f_n''}(\zeta_n) = 0$ then H_1 must be holomorphic function in $\Delta(0,r)$. Differentiation then gives

(5) $\quad 1 - \frac{f_n' f_n''}{(f_n'')^2} \Rightarrow H_1'$ in $\Delta(0,r)$.

At $z = s_n$ the left hand of (5) is equal to 1. on the other hand in some small neighbourhood of $z = \zeta_n$, We have $f_n(z) = \frac{A_n}{z-z_n} + \hat{f}_n(z)$, where $A_n \neq 0$ is a constant, and $\hat{f}_n(z)$ is analytic. Here we used that according to the assumption of Theorem 1 , all poles of f_n must be simple.

Hence we have $f_n'(z) = \frac{-A_n}{(z-\zeta_n)^2} + \hat{f}_n'(z), f_n''(z) = \frac{2A_n}{(z-\zeta_n)^3} + \hat{f}_n''(z), f_n^{(3)}(z) = \frac{-6A_n}{(z-z_n)^4} + \hat{f}_n^{(3)}(z)$. Then the left hand of (5) get at $z = \zeta_n$. The value $1 - \frac{6}{4} = -\frac{1}{2} \neq 1$, a contradiction .

Claim there exist $r > 0$ and $k > 0$ such that for large enough n, $|\frac{f_n}{f_n''}(z)|$, $\left|\frac{f_n^2}{f_n''}(z)\right| \leq K$ for $z \in \Delta(0,r)$.

Proof of Claim Since $H(0) = \infty$,there exist $r > 0$ and $M > 0$ such that $\overline{\Delta}(0,r) \subset D$ and such that for large enough n, $|f_n''(z)| > M$ for $z \in \Delta(0,r)$.

Now ,if $|f_n(z)| \leq |f_n''(z)|^{\frac{1}{3}}$ then

(6) $\quad |\frac{f_n}{f_n''}(z)| \leq \frac{|f_n''(z)|^{\frac{1}{3}}}{|f_n''(z)|} \leq \frac{1}{M^{\frac{2}{3}}}$

and

(7) $\quad |\frac{f_n^2}{f_n''}(z)| \leq \frac{|f_n''(z)|^{\frac{2}{3}}}{|f_n''(z)|} \leq \frac{1}{M^{\frac{1}{3}}}$.

If on the other hand $|f_n(z)| \geq |f_n''(z)|^{\frac{1}{3}}$, then since $\frac{x}{1+x^3} \leq \frac{2^{\frac{2}{3}}}{3}$ for $x \geq 0$, we get

(8) $\quad |\frac{f_n}{f_n''}(z)| = \frac{1+|f_n(z)|^3}{|f_n''(z)|} \cdot \frac{|f_n(z)|}{1+|f_n(z)|^3} \leq \frac{1}{C} \cdot \frac{2^{\frac{2}{3}}}{3}$.

Also We have $\frac{x^2}{1+x^3} \leq \frac{2^{\frac{2}{3}}}{3}$ for $x \geq 0$ and thus

(9) $\quad |\frac{f_n^2}{f_n''}(z)| = \frac{1+|f_n(z)|^3}{|f_n''(z)|} \cdot \frac{|f_n^2(z)|}{1+|f_n(z)|^3} \leq \frac{1}{C} \cdot \frac{2^{\frac{2}{3}}}{3}$.

The claim then follows by taking $k = \max\{\frac{1}{M^{\frac{2}{3}}}, \frac{1}{M^{\frac{1}{3}}}, \frac{1}{C} \cdot \frac{2^{\frac{2}{3}}}{3}\}$ and consider (6),(7),(8)and (9).

From the claim we deduce that $\{\frac{f_n}{f_n''}\}_{=1}^{\infty}$ and $\{\frac{f_n^2}{f_n''}\}_{=1}^{\infty}$ are normal in $\Delta(0,r)$, so after moving to a subsequence, that also will be denote by $\{f_n\}_{n=1}^{\infty}$, we get that

(10) $\qquad \dfrac{f_n}{f_n''} \to H_1$ in $\Delta(0,r)$

and

(11) $\qquad \dfrac{f_n^2}{f_n''} \to H_2$ in $\Delta(0,r)$

From the claim it follows that H_1 and H_2 are holomorphic in $\Delta(0,r)$.

Differentiating (10) and (11) gives respectively

(12) $\qquad \dfrac{f_n'}{f_n''} - \dfrac{f_n^{(3)}}{f_n''} \cdot f_n \Rightarrow H_1'$ in $\Delta(0,r)$

and

(13) $\qquad 2f_n \cdot \dfrac{f_n'}{f_n''} - f_n^2 \cdot \dfrac{f_n^{(3)}}{f_n''^2} \Rightarrow H_2'$ in $\Delta(0,r)$.

Since $\{f_n''\}_{n=1}^{\infty}$ is normal, there exists some $k_1 > 0$ such that $\frac{|f_n^{(3)}(z)|}{1+|f_n''(z)|} \le k_1$ for every $n \ge 1$ and for every $z \in \Delta(0,r)$. Since in addition for large enough n, $|f_n''(z)| > M$, then

$$\begin{aligned}\frac{|f_n^{(3)}(z)|}{|f_n''(z)|^2} &= \frac{|f_n^{(3)}(z)|}{1+|f_n''(z)|^2}\frac{1+|f_n''(z)|^2}{|f_n''(z)|^2} \\ &\le k_1(1+\frac{1}{M^2}) := k_2.\end{aligned}$$

Thus

(14) $\qquad \frac{|f_n''(z)|^2}{|f_n^{(3)}(z)|} \ge \frac{1}{k^2}$ for large enough n.

Now since we assume that $\{\frac{f_n'}{f_n''}\}$ is not normal at $z=0$, then after moving to a subsequence, that also will be denoted by $\{f_n\}_{n=1}^{\infty}$, we get that there exists a sequence of points $t_n \to 0$, such that

$$\frac{f_n'}{f_n''}(t_n) := M_n \to \infty, \quad M_n \in \mathbb{C}.$$

Substituting $z = t_n$ in (12) gives

(15)

$$M_n - \frac{f_n^{(3)} \cdot f_n}{f_n''^2}(t_n) := \varepsilon_n \to H_1'(0).$$

Hence

$$f_n(t_n) = (M_n - \varepsilon_n)\frac{f_n''^2}{f_n^{(3)}}(t_n)$$

From (15) we get, by substituting $z = t_n$ in (13)

$$2(M_n - \varepsilon_n)\frac{f_n''^2}{f_n^{(3)}}(t_n)M_n - (M_n - \varepsilon_n)^2 \left(\frac{f_n''^2}{f_n^{(3)}}(t_n)\right)^2 \frac{f_n^{(3)}}{f_n''^2}(t_n) := \delta_n \to H_2'(0).$$

From this we get after simplifying

$$(M_n^2 - \varepsilon_n^2)\frac{f_n''^2}{f_n^{(3)}}(t_n) = \delta_n.$$

But by (14) the left hand above tends to ∞ as $n \to \infty$, while the right hand is bounded, a contradiction.

This completes the proof of Theorem 1.

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